24 maart 2011

1 ~~$a_n' = n \cdot f'(\tfrac{1}{n}) \cdot \tfrac{1}{n} = f'(\tfrac{1}{n})$~~

2 $\sum_{n=1}^{\infty} \frac{(x-2)^{n+1}}{3^n \sqrt{n}}$

$$: \lim_{n\to\infty} \left|\frac{a_{n+1}}{a_n}\right| = \lim_{n\to\infty} \left|\frac{(x-2)^{n+2}}{3^{n+1}\sqrt{n+1}}\right| \cdot \left|\frac{3^n \sqrt{n}}{(x-2)^{n+1}}\right| = \lim_{n\to\infty} \left|\frac{(x-2)^{n+1}(x-2)}{(x-2)^{n+1}}\right| \cdot \left|\frac{3^n \cdot \sqrt{n}}{3^n \cdot 3 \cdot \sqrt{n+1}}\right|$$

$$= \lim_{n\to\infty} \left|\frac{(x-2)\sqrt{n}}{3 \cdot \sqrt{n+1}}\right| = \lim_{n\to\infty} \left|\frac{x-2}{3}\right| \cdot \lim_{n\to\infty} \left|\frac{\sqrt{n}}{\sqrt{n+1}}\right| = \lim_{n\to\infty} \left|\frac{x-2}{3}\right| = \left|\frac{x-2}{3}\right|$$

Dit is kleiner dan 1 als $\left|\frac{x-2}{3}\right| < 1$

$|x-2| < 3$, dus $x \langle -1, 5 \rangle$ Dan is de reeks absoluut convergent.

~~rondt~~

~~als $x \in \mathbb{R}$~~ als $x < -1$ of $x > 5$, dan $\left|\frac{x-2}{3}\right| > 1$, dus dan is de reeks divergent.

Voor de grenswaarde $x = -1$ geldt: $\sum_{n=1}^{\infty} \frac{(-1-2)^{n+1}}{3^n \sqrt{n}} = \sum_{n=1}^{\infty} (-1)^{n+1} \frac{3^{n+1}}{3^n \sqrt{n}} = \sum_{n=1}^{\infty} (-1)^{n+1} \frac{3}{\sqrt{n}}$.

Deze is niet absoluut convergent:

$\sum_{n=1}^{\infty} \left|(-1)^{n+1} \frac{3}{\sqrt{n}}\right| = \sum_{n=1}^{\infty} \frac{3}{\sqrt{n}} = 3 \sum_{n=1}^{\infty} \frac{1}{\sqrt{n}}$, een p reeks met $|p| < 1$ dus divergent.

Wel voorwaardelijk, want:

* $\frac{3}{\sqrt{n}} > \frac{3}{\sqrt{n+1}}$ voor n 1, 2, ... dus de rij is dalend
* $\lim_{n\to\infty} \frac{3}{\sqrt{n}} = 0$

$\Rightarrow$ alternating series test zegt convergent.

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Voor grenswaarde $x = 5$ geldt: $\sum_{n=1}^{\infty} \frac{(5-2)^{n+1}}{3^n \sqrt{n}} = \sum_{n=1}^{\infty} 3 \cdot \frac{3^n}{3^n \sqrt{n}} = 3 \sum_{n=1}^{\infty} \frac{1}{\sqrt{n}}$, een p reeks met $|p| < 1$, dus divergent.

Samenvattend:

divergent voor $x < -1$ en $x \geq 5$

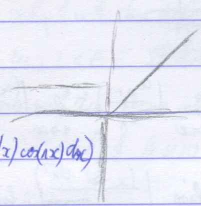
Conditioneel convergent voor $x = -1$

absoluut convergent voor $-1 < x < 5$

3a $a_0 = \frac{1}{2\pi}\int_{-\pi}^{\pi} f(x)\,dx$

$a_n = \frac{1}{\pi}\int_{-\pi}^{\pi} f(x)\cos(nx)\,dx$

$b_n = \frac{1}{\pi}\int_{-\pi}^{\pi} f(x)\sin(nx)\,dx$



$a_n = \frac{1}{\pi}\int_{-\pi}^{\pi} f(x)\cos(nx)\,dx = \frac{1}{\pi}\left(\int_{-\pi}^{0} f(x)\cos(nx)\,dx + \int_{0}^{\pi} f(x)\cos(nx)\,dx\right)$

$= \frac{1}{\pi}\left(\int_{-\pi}^{0} 1\cdot\cos(nx)\,dx + \int_{0}^{\pi} x\cos(nx)\,dx\right)$

$= \frac{1}{\pi}\left(\left[\frac{1}{n}\sin(nx)\right]_{-\pi}^{0} + \left[\frac{1}{n}\sin(nx)\cdot x\right]_{0}^{\pi} - \frac{1}{n}\int_{0}^{\pi}\sin nx\right)$

$= \frac{1}{n\pi}\left(\;(0) + (0) + \left[\frac{1}{n}\cos nx\right]_{0}^{\pi}\right)$

$= \begin{cases} \frac{1}{n^2\pi}\cdot -2 & \text{als } n \text{ oneven} \\ 0 & \text{als } n \text{ even} \end{cases}$

$a_0 = \frac{1}{2\pi}\int_{-\pi}^{\pi} f(x)\,dx = \frac{1}{2\pi}\left(\int_{-\pi}^{0} f(x)\,dx + \int_{0}^{\pi} f(x)\,dx\right) = \frac{1}{2\pi}\left(\int_{-\pi}^{0} 1\,dx + \int_{0}^{\pi} x\,dx\right)$

$= \frac{1}{2\pi}\left([x]_{-\pi}^{0} + \left[\frac{1}{2}x^2\right]_{0}^{\pi}\right) = \frac{1}{2\pi}\left((\pi) + \left(\frac{1}{2}\pi^2\right)\right) = \frac{1+\frac{1}{2}\pi}{2}$

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$b_n = \frac{1}{\pi}\int_{-\pi}^{\pi} f(x)\sin(nx)\,dx = \frac{1}{\pi}\left(\int_{-\pi}^{0} f(x)\sin(nx)\,dx + \int_{0}^{\pi} f(x)\sin(nx)\,dx\right) = \frac{1}{\pi}\left(\int_{-\pi}^{0}\sin(nx)\,dx + \int_{0}^{\pi} x\sin(nx)\,dx\right)$

$= \frac{1}{\pi}\left(\left[-\frac{1}{n}\cos nx\right]_{-\pi}^{0} + \left[-\frac{x}{n}\cos nx\right]_{0}^{\pi} + \frac{1}{n}\int_{0}^{\pi}\cos nx\right) = \frac{-1}{n\pi}\left([\cos nx]_{-\pi}^{0} + [\cos nx\cdot x]_{0}^{\pi} - \left[\frac{1}{n}\sin nx\right]_{0}^{\pi}\right)$

$= \frac{-1}{n\pi}\left([\cos nx]_{-\pi}^{0} + [\cos nx\cdot x]_{0}^{\pi} + 0\right)$

$= \frac{-1}{n\pi}\left((1-\cos(n\pi)) + (\pi\cos(n\pi) - 0)\right) = \frac{-1}{n\pi}\left(1+\cos(n\pi)(\pi-1)\right)$

7 $= \begin{cases} \frac{-1}{n\pi}(1-(\pi-1)) = \frac{\pi-1-1}{n\pi} = \frac{\pi-2}{n\pi} & \text{als } n \text{ oneven} \\ \frac{-1}{n\pi}(1+(\pi-1)) = \frac{-(-\pi+1}{n\pi} = \frac{-\pi}{n\pi} = -\frac{1}{n} & \text{als } n \text{ even} \end{cases}$

Van V.

dus $f(x) = \frac{1+\frac{1}{2}\pi}{2} + \left(\frac{-2}{\pi}\cdot\cos x + \frac{\pi-2}{\pi}\cdot\sin x\right) + \left(-\frac{1}{2}\cdot\sin 2x\right) + \left(\frac{-2}{9\pi}\cdot\cos 3x + \frac{\pi-2}{3\pi}\cdot\sin 3x\right) + \left(-\frac{1}{4}\sin 4x\right)\ldots$

b ~~Dat is een niet continu~~ Daar is $f(x)$ niet continu, dus neemt de Fourierreeks het gemiddelde van de linker en rechterlimiet aan, dus $\left(\lim_{x\to 0^+} f(x) + \lim_{x\to 0^-} f(x)\right)/2 = \frac{0+1}{2} = \frac{1}{2}$

4

4 a $\int_0^{2\pi} \sqrt{\left(\frac{dx}{dt}\right)^2 + \left(\frac{dy}{dt}\right)^2}\, dt = \int_0^{2\pi} \sqrt{(-\sin t + \sin t + t\cos t)^2 + (\cos t - \cos t + t \sin t)^2}\, dt = \int_0^{2\pi} \sqrt{t^2\cos^2 t + t^2 \sin^2 t}\, dt$

$= \int_0^{2\pi} t\sqrt{\cos^2 t + \sin^2 t}\, dt = \int_0^{2\pi} t\sqrt{1}\, dt = \int_0^{2\pi} t\, dt = \left[\frac{1}{2}t^2\right]_0^{2\pi} = \frac{1}{2}\cdot(2\pi)^2 = 2\pi^2$

b $\frac{dy}{dx} = \frac{dy/dt}{dx/dt} = \frac{t \sin t}{t \cos t} = \frac{\sin t}{\cos t} = \tan t$ mits $t \neq 0$ en $\cos t \neq 0$.

10

$\frac{dy}{dx}$ als $t = \frac{\pi}{4}$: $\tan\frac{\pi}{4} = \frac{\sin\frac{\pi}{4}}{\cos\frac{\pi}{4}} = \frac{\frac{1}{2}\sqrt{2}}{\frac{1}{2}\sqrt{2}} = 1$

$\frac{dy}{dx}$ als $t = \pi$: $\tan \pi = \frac{\sin\pi}{\cos\pi} = \frac{0}{-1} = 0$

5 a $P(x) = 1 + \frac{f'(0)}{1!}\cdot x + \frac{f''(0)}{2!}\cdot x^2 + \frac{f'''(0)}{3!}\cdot x^3 \ldots$

$= 1 + \frac{1}{1}\cdot x + \frac{2}{2!}\cdot x^2 + \frac{0}{3!}\cdot x^3 + 0 + 0 + 0 \ldots$

$= 1 + x + x^2$

b $P(x) = \frac{f(a)}{0!}\cdot(x-a)^0 + \frac{f'(a)}{1!}\cdot(x-a)^1 + \frac{f''(a)}{2!}\cdot(x-a)^2 + \frac{f'''(a)}{3!}\cdot(x-a)^3 \ldots$

$= \frac{3}{1} + \frac{3}{1}\cdot(x-1) + \frac{2}{2}\cdot(x-1)^2 + \frac{0}{3!}\cdot(x-a)^3 + 0 + 0 \ldots$

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$= 3 + 3(x-1) + (x-1)^2 = 3 + 3x - 3 + x^2 - 2x + 1 = 1 + x + x^2$

6 a Ja, hij convergeert. Bewijs:

$\sum_{n=1}^{\infty} a_n$ is convergent $\Rightarrow$ $\lim_{n\to\infty} a_n = 0$. Dan volgt uit de definitie van limiet dat $\forall \varepsilon > 0$, $\exists N$ waarvoor als $n > N$, dan $|a_n| < \varepsilon$. Dus, als $\varepsilon = 1$, dan is a_n vanaf de N^e term ≤ 1. a_n^2 is vanaf die term kleiner dan a_n, want een getal groter dan 0, kleiner ~~dan~~ of gelijk aan 1 is in het kwadraat kleiner ~~dan~~ of gelijk aan zichzelf.

Daaruit volgd (met gebruik van inklemming tussen a_n en 0) dat

10 $\sum_{n=N}^{\infty} a_n^2 \leq \sum_{n=N}^{\infty} a_n \Rightarrow \sum_{n=N}^{\infty} a_n^2$ is convergent.

$\sum_{n=1}^{N} a_n^2$ is een eindig getal dus convergent

$\sum_{n=1}^{\infty} a_n^2 = \sum_{n=1}^{N} a_n^2 + \sum_{n=N}^{\infty} a_n^2$

$\Rightarrow \sum_{n=1}^{\infty} a_n^2$ is convergent.

b a_n convergeert naar 0 want $\sum_{n=1}^{\infty} a_n$ is convergent.

$1-a_n$ convergeerd dus naar 1.

$\lim_{n\to\infty} \frac{a_n}{1-a_n} = 0$

1 Omdat b_n het quotient is van 2 convergende rijen die in de limiet samen ~~$\sum_{n=1}^{\infty} b_n$ ook convergeren op basis van de comparison test~~ 0 opleveren moet $\sum_{n=1}^{\infty} b_n$ ook convergeren op basis van de comparison test.

1

$\lim_{n\to\infty} n f\left(\frac{1}{n}\right) = \lim_{n\to\infty} \frac{f\left(\frac{1}{n}\right)}{\frac{1}{n}}$, teller en noemer gaan naar 0, dus l'Hospital

geeft $\lim_{n\to\infty} \frac{f'\left(\frac{1}{n}\right) \cdot \frac{-1}{n^2}}{-\frac{1}{n^2}} = \lim_{n\to\infty} f'\left(\frac{1}{n}\right) = f'(0)$

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